Loan Guarantees
Part I - The Two State Guarantee Model

Gary Schurman MBE, CFA

April, 2017

We will define enterprise value to be company value excluding debt. We want to value a loan guarantee using a simple two state model where enterprise value at time t can be one of two values depending on the state of the world at that time. We will use this model to construct a hedge for a short position in a loan guarantee. To assist us in our endeavors we will use the following hypothetical problem throughout this series...

Our Hypothetical Problem

We are tasked with building a two state model where the states of the world at time t are [Default] and [No Default]. Annualized cash flow excluding debt service is \$100,000, the cash flow growth rate is 2.50% and the cost of capital is 10.00%. The company is contractually obligated to pay of it's zero-coupon debt at the end of year three where the scheduled debt payoff amount is \$500,000. The probability of default is 10%, the debt recovery rate given default is 40%, the risk-free rate is 4.00%, and the payoff on a three year, risk-free, zero-coupon bond is \$100,000.

Questions:
1) What are enterprise value and accumulated cash value at the end of year three assuming no default?
2) What are enterprise value and accumulated cash value at the end of year three assuming default?
3) What is the guarantor's obligation under the terms of the guarantee given default?
4) What is the value of the loan guarantee at time zero?

Building The Model

We will define the variable A_t to be enterprise value at time t, the random variable $N(t)$ to be a jump counter that counts the number of jumps that occur over the time interval $[0, t]$, the variable λ to be the drift parameter, the variable ψ to be the jump parameter, and the variable ω to be jump size. The differential equation that defines how enterprise value changes over time is...

$$\delta A_t = \lambda\, A_t\, \delta t + \psi\, A_t\, \delta N(t) \text{ ...where... } \delta N(t) \in [0,1] \text{ ...and... } \psi = \ln(1+\omega) \tag{1}$$

The solution to Equation (1) above is the equation for enterprise value at time t, which is...

$$A_t = (1+\omega)^{N(t)} A_0 \,\mathrm{Exp}\left\{\lambda\, t\right\} \text{ ...where... } N(0)=0 \text{ ...and... } N(t) \in [0,1] \tag{2}$$

Note that per Equation (2) above the number of jumps at time t can be either zero or one and therefore there are two possible states of the world at time t. We will define enterprise value at time zero to be the value assigned to the enterprise per the valuation report. If we define the variable C_0 to be annualized cash flow excluding debt service at time zero, the variable g to be the discrete time cash flow growth rate, and the variable r to be the discrete time cost of capital then the equation for enterprise value at time zero using the dividend discount model is...

$$A_0 = \sum_{t=1}^{\infty} C_0 \left(\frac{1+g}{1+r}\right)^t = \frac{C_0\,(1+g)}{r-g} \tag{3}$$

Using Equation (3) above and the parameters to our hypthetical problem, enterprise value at time zero is...

$$A_0 = \frac{100{,}000 \times (1+0.0250)}{0.1000-0.0250} = 1{,}366{,}700 \tag{4}$$

We will define the variable μ to be the continuous time cash flow growth rate and the variable κ to be the continuous time discount rate. The equation for enterprise value at time zero using these continuous time model parameters is...

$$A_0 = \int_0^\infty C_0 \,\mathrm{Exp}\left\{\mu\, t\right\} \mathrm{Exp}\left\{-\kappa\, t\right\} \delta t = \frac{C_0}{\kappa - \mu} \tag{5}$$

We will define the variable ϕ to be the continuous time dividend yield. Using Equations (3) and (5) above we will make the following definitions...

$$\text{if...}\;\; \mu = \ln(1+g) \;\;\text{...then...}\;\; \kappa = \frac{C_0}{A_0} + \mu \;\;\text{...and...}\;\; \phi = \kappa - \mu \tag{6}$$

Using Equation (6) above and the parameters to our hypothetical problem, the values of the continuous time model parameters are...

$$\mu = \ln(1 + 0.0250) = 0.0247 \;\;;\;\; \kappa = \frac{100,000}{1,366,700} + 0.0247 = 0.0979 \;\;;\;\; \phi = 0.0979 - 0.0247 = 0.0732 \tag{7}$$

Jump Probability

We are currently standing at time zero and we want to find the probability that a jump will occur at some time over the infinitesimally small time interval $[t, t+\delta t]$. We will define the variable τ to be the arrival time of the next jump and Δ to be jump intensity, which we will define as the average number of jumps that occur in any one year. If we assume that the time between jumps is exponentially distributed then the equation for jump probability is... [1]

$$P\left[t < \tau \le t + \delta t\right] = \Delta \,\mathrm{Exp}\left\{-\Delta\, t\right\} \delta t \;\;\text{...where...}\;\; u \ge 0 \tag{8}$$

Using Equation (8) above and Appendix Equation (50) below, the probability that a jump will occur over the time interval $[0, t]$ is...

$$P\left[\tau \le t\right] = \int_0^t \Delta \,\mathrm{Exp}\left\{-\Delta\, s\right\} \delta s = 1 - \mathrm{Exp}\left\{-\Delta\, t\right\} \tag{9}$$

Using Equation (9) above the probability that a jump will not occur over the time interval $[0, t]$ is...

$$P\left[\tau > t\right] = 1 - P\left[\tau \le t\right] = \mathrm{Exp}\left\{-\Delta\, t\right\} \tag{10}$$

We will assume that if there is a jump over the time interval $[0, t]$ then enterprise value at time t is less than the debt payoff amount at time t and therefore the company defaults on the debt. We will define the variable p to be the cumulative probability of default over the time interval $[0, t]$. Using Equations (9) and (10) above the equations for the probability of default and the probability of no default over the time interval $[0, t]$ are...

$$P\left[\tau \le t\right] = 1 - \mathrm{Exp}\left\{-\Delta\, t\right\} = p \;\;\text{...and...}\;\; P\left[\tau > t\right] = \mathrm{Exp}\left\{-\Delta\, t\right\} = 1 - p \tag{11}$$

Using Equation (11) above we can solve for the jump intensity parameter Δ as follows...

$$\text{if...}\;\; P\left[\tau \le t\right] = p \;\;\text{...then...}\;\; p = 1 - \mathrm{Exp}\left\{-\Delta\, t\right\} \;\;\text{...such that...}\;\; \Delta = -\frac{\ln(1-p)}{t} \tag{12}$$

Using Equation (12) above and the parameters to our hypthetical problem, the value of the model parameter Δ is...

$$\Delta = -\frac{\ln(1 - 0.10)}{3.00} = 0.0351 \;\;\text{...given that...}\;\; t = \text{debt maturity in years} = 3.00 \tag{13}$$

Equations For Debt Value

We will define the variable D_t to be the scheduled debt payoff amount at time t and the variable π to be the debt recovery rate given default. If the jump did not occur over the time interval $[0, t]$ then the equation for debt value at time t from the perspective of time zero is...

$$\left[D_t \middle| \tau > t\right] = D_t \tag{14}$$

If the jump did occur over the time interval $[0, t]$ then the equation for debt value at time t from the perspective of time zero is...

$$\left[D_t \middle| \tau \leq t\right] = \pi \, D_t \tag{15}$$

Equations For Enterprise Value

Using Equation (2) above if the jump did not occur over the time interval $[0, t]$ then the equation for enterprise value at time t from the perspective of time zero is...

$$\left[A_t \middle| \tau > t\right] = A_0 \operatorname{Exp}\left\{\lambda \, t\right\} \text{ ...because... } N(t) = 0 \tag{16}$$

Using Equation (2) above if the jump did occur over the time interval $[0, t]$ then the equation for asset value at time t from the perspective of time zero is...

$$\left[A_t \middle| \tau \leq t\right] = (1 + \omega) \, A_0 \operatorname{Exp}\left\{\lambda \, t\right\} \text{ ...because... } N(t) = 1 \tag{17}$$

Using Equations (11), (16) and (17) above and the parameters to our hypothetical problem the equation for expected asset value at time t from the perspective of time zero is...

$$\mathbb{E}\left[A_t\right] = (1 - p) \left[A_t \middle| \tau > t\right] + p \left[A_t \middle| \tau \leq t\right] \tag{18}$$

Note that the differential equation that underlies the dividend discount model (Equation (3) above) implies that the equation for expected enterprise value at time t can also be written as...

$$\mathbb{E}\left[A_t\right] = A_0 \operatorname{Exp}\left\{\mu \, t\right\} \tag{19}$$

If we equate Equations (18) and (19) above then we get the following equation...

$$A_0 \operatorname{Exp}\left\{\mu \, t\right\} = (1 - p) \left[A_t \middle| \tau > t\right] + p \left[A_t \middle| \tau \leq t\right] \tag{20}$$

In order to use asset value Equation (2) above we need values for model parameters λ and ω. We want to set asset value at time t given default equal to the liquidation value of the enterprise (i.e. the expected recovery on the debt given default). Using Equations (15) and (16) above we can rewrite Equation (20) above as...

$$A_0 \operatorname{Exp}\left\{\mu \, t\right\} = (1 - p) \, A_0 \operatorname{Exp}\left\{\lambda \, t\right\} + p \, \pi \, D_t \tag{21}$$

Using Equation (21) above and solving for λ the equation for the model parameter λ is...

$$\lambda = \ln\left[\left(A_0 \operatorname{Exp}\left\{\mu \, t\right\} - p \, \pi \, D_t\right) \Big/ ((1 - p) \, A_0)\right] \Big/ t \tag{22}$$

Using Equation (22) above and the parameters to our hypothetical problem, the value of model parameter λ given that t = debt maturity in years = 3.00 and p = cumulative probability of default = 0.10 is...

$$\lambda = \ln\left[\left(1,366,700 \times \operatorname{Exp}\left\{0.0247 \times 3\right\} - 0.10 \times 0.40 \times 500,000\right) \Big/ (0.90 \times 1,366,700)\right] \Big/ 3 = 0.0553 \tag{23}$$

Since we set enterprise value given default equal to the liquidation value of the enterprise then using Equations (15) and (17) above we can solve for the model parameters ω as follows...

$$\text{if... } \pi\, D_t = (1+\omega)\, A_0 \,\text{Exp}\left\{\lambda\, t\right\} \text{ ...then... } \omega = \frac{\pi\, D_t}{A_0} \,\text{Exp}\left\{-\lambda\, t\right\} - 1 \tag{24}$$

Using Equations (23) and (24) above and the parameters to our hypothetical problem, the value of model parameter ω is...

$$\omega = \frac{0.40 \times 500,000}{1,366,700} \times \text{Exp}\left\{-0.0553 \times 3\right\} - 1 = -0.8760 \tag{25}$$

Equation For Bank Account Value

We will define the variable B_t to be bank account value at time t and the variable α to be the risk-free interest rate. Dividends paid by the enterprise are deposited into the bank account and earn interest at the risk-free rate. Using the parameters to our hypothetical problem the equation for the continuous time risk-free rate is...

$$\alpha = \ln(1+0.04) = 0.0392 \tag{26}$$

We will define the variable $\hat{\mu}$ to be the average cash flow growth rate, which depends on if there was a jump over the time interval $[0,t]$. We will define the variable A_t^N to be asset value at time t given that there wasn't a jump and the variable A_t^D to be asset value at time t given that there was a jump. The equation for the average cash flow growth rate is...

$$\left[\hat{\mu}\,\middle|\,\tau > t\right] = \ln\left(\frac{A_t^N}{A_0}\right)\frac{1}{t} \text{ ...and... } \left[\hat{\mu}\,\middle|\,\tau \le t\right] = \ln\left(\frac{A_t^D}{A_0}\right)\frac{1}{t} \tag{27}$$

Using Equations (26) and (27) above and the parameters to our hypothetical problem the equation for bank account value at the end of time T is...

$$B_t = \int_0^t C_0 \,\text{Exp}\left\{\hat{\mu}\, s\right\} \text{Exp}\left\{\alpha\,(t-s)\right\} \delta s = C_0 \,\text{Exp}\left\{\alpha\, t\right\} \int_0^t \text{Exp}\left\{(\hat{\mu}-\alpha)\, s\right\} \delta s \tag{28}$$

The solution to Equation (28) above is...

$$B_s = C_0 \,\text{Exp}\left\{\alpha\, t\right\} \left(\text{Exp}\left\{(\hat{\mu}-\alpha)\, t\right\} - 1\right) \left(\hat{\mu}-\alpha\right)^{-1} \tag{29}$$

Equation For Risk-Free Bond Value

We will define the variable M_t to be value of a zero-coupon, risk-free bond at time t. The equation for bond value at time zero is...

$$M_0 = M_t \,\text{Exp}\left\{-\alpha\, t\right\} \tag{30}$$

Using Equations (26) and (30) above and the parameters to our hypothetical problem, the value of the zero-coupon, risk-free bond at time zero is...

$$M_0 = 100,000 \times \text{Exp}\left\{-0.0392 \times 3.00\right\} = 88,900 \text{ ...given that... } t = \text{debt maturity in years} = 3.00 \tag{31}$$

We will define the variable M_t^N to be risk-free bond value at time t given that there wasn't a jump and the variable M_t^D to be risk-free bond value at time t given that there was a jump. Using the parameters to our hypothetical problem the equation for the risk-free bond value at time t is...

$$M_t^N = M_t^D = 100,000 \tag{32}$$

Equation For Guarantee Value

We want hedge the Gurantor's risk under the terms of the guarantee. We will define the variable G_0 to be guarantee value at time zero, the variable G_t^N to be guarantee value at time t given no default, and the variable G_t^D to be guarantee value at time t given default. The following table presents the payoffs on the enterprise, risk-free bond and guarantee in our two state world ...

Table 1: State Payoffs

Asset Class	Price Today	Payoffs at Time t: No Default	Payoffs at Time t: Default
Enterprise	A_0	A_t^N	A_t^D
Risk-free bond	M_0	M_t^N	M_t^D
Guarantee	G_0	G_t^N	G_t^D

We will define the variables U_A and U_M to be the units of the enterprise and the risk-free bond, respectively, that we either long or short. We want to build a hedge that satisfies the following two equations...

$$\text{No Default at time } t\text{: } U_A \times A_t^N + U_M \times M_t^N = G_t^N$$
$$\text{Default at time } t\text{: } U_A \times A_t^D + U_M \times M_t^D = G_t^D \tag{33}$$

We will define the following matrix and vectors...

$$\mathbf{A} = \begin{bmatrix} A_t^N & M_t^N \\ A_t^D & M_t^D \end{bmatrix} \quad \vec{\mathbf{v}} = \begin{bmatrix} U_A \\ U_M \end{bmatrix} \quad \vec{\mathbf{u}} = \begin{bmatrix} G_t^N \\ G_t^D \end{bmatrix} \tag{34}$$

Using Equation (34) above we can rewrite Equation (33) above via the following matrix:vector product...

$$\mathbf{A}\vec{\mathbf{v}} = \vec{\mathbf{u}} \tag{35}$$

Using Equation (35) above the solution to vector $\vec{\mathbf{v}}$, which is the vector of units to long or short, is...

$$\vec{\mathbf{v}} = \mathbf{A}^{-1}\vec{\mathbf{u}} \tag{36}$$

Using the vector of units (see Equation (36) above) and asset prices at time zero (see Table 1 above) the equation for the value of the guarantee at time zero is...

$$G_0 = U_A \times A_0 + U_M \times M_0 \tag{37}$$

The Solution To Our Hypothetical Problem

Using the equations above the following table presents the model parameters...

Table 2: Model Parameters

Symbol	Description	Value	Reference
A_0	Enterprise value at time zero	1,366,700	Equation (4)
C_0	Annualized cash flow at time zero	100,000	Hypothetical problem
D_T	Debt payoff amount at time T	500,000	Hypothetical problem
M_0	Risk-free bond value at time zero	88,900	Equation (31)
T	Guarantee term in years	3.0000	Hypothetical problem
α	Risk-free rate (continuous time)	0.0392	Equation (26)
κ	Cost of capital (continuous time)	0.0979	Equation (7)
μ	Cash flow growth rate (continuous time)	0.0247	Equation (7)
λ	Enterprise value equation drift parameter	0.0553	Equation (23)
ω	Enterprise value equation jump size	-0.8760	Equation (25)
p	Cumulative probability of default	0.1000	Hypothetical problem
ϕ	Dividend yield (continuous time)	0.0732	Equation (7)
π	Recovery rate given default	0.4000	Hypothetical problem

Answer to Question 1: Using Equation (16) above and the parameters from Table 2 above enterprise value at the end of year three assuming no default is...

$$\left[A_T \,\middle|\, \tau > T\right] = 1,366,700 \times \text{Exp}\left\{0.0553 \times 3.00\right\} = 1,613,100 \tag{38}$$

Using Equation (27) above the average growth rate over the time interval $[0, T]$ assuming no default is...

$$\left[\hat{\mu} \,\middle|\, \tau > T\right] = \ln\left(\frac{1,613,100}{1,366,700}\right) \Big/ 3 = 0.0553 \tag{39}$$

Using Equation (29) above bank account value at the end of year three assuming no default is...

$$\left[B_T \,\middle|\, \tau > T\right] = 100,000 \times \text{Exp}\left\{0.0392 \times 3\right\}\left(\text{Exp}\left\{(0.0553 - 0.0392) \times 3\right\} - 1\right)\left(0.0553 - 0.0392\right)^{-1} = 345,700 \tag{40}$$

Answer to Question 2: Using Equation (16) above and the parameters from Table 2 above enterprise value at the end of year three assuming default is...

$$\left[A_T \,\middle|\, \tau > T\right] = (1 - 0.8760) \times 1,366,700 \times \text{Exp}\left\{0.0553 \times 3\right\} = 200,000 \tag{41}$$

Using Equation (27) above the average growth rate over the time interval $[0, T]$ assuming default is...

$$\left[\hat{\mu} \,\middle|\, \tau > T\right] = \ln\left(\frac{200,000}{1,366,700}\right) \Big/ 3 = -0.6406 \tag{42}$$

Using Equation (29) above bank account value at the end of year three assuming default is...

$$\left[B_T \,\middle|\, \tau > T\right] = 100,000 \times \text{Exp}\left\{0.0392 \times 3\right\}\left(\text{Exp}\left\{(-0.6406 - 0.0392) \times 3\right\} - 1\right)\left(-0.6460 - 0.0392\right)^{-1} = 143,900 \tag{43}$$

Answer to Question 3: What is the guarantor's obligation under the terms of the guarantee given default?

$$G_T^D = D_T - A_T^D = 500,000 - 200,000 = 300,000 \tag{44}$$

Answer to Question 4: What is the value at time zero of the loan guarantee?

We will define asset value at time T to be enterprise value at time T plus bank account value at time T. Using the answers to questions one and two above total asset value at time T is...

$$\text{No default: } 1,613,100 + 345,700 = 1,958,800 \text{ ...and... Default: } 200,000 + 143,900 = 343,900 \tag{45}$$

Using Equations (31) and (34) above and the answers to questions one, two and three above the payoff matrix and vector definitions are...

$$\mathbf{A} = \begin{bmatrix} 1,958,800 & 100,000 \\ 343,900 & 100,000 \end{bmatrix} \quad \vec{\mathbf{v}} = \begin{bmatrix} U_A \\ U_M \end{bmatrix} \quad \vec{\mathbf{u}} = \begin{bmatrix} 0 \\ 300,000 \end{bmatrix} \tag{46}$$

Using Equations (36) and (46) above the vector of units is...

$$\vec{\mathbf{v}} = \mathbf{A}^{-1}\vec{\mathbf{u}} = \begin{bmatrix} -0.1858 \\ 3.6389 \end{bmatrix} \tag{47}$$

Using Equation (37) and (47) above the value of the guarantee at time zero is...

$$G_0 = -0.1858 \times 1,366,700 + 3.6389 \times 88,900 = 69,600 \tag{48}$$

Note for Excel Users: If matrix $\mathbf{A}$ is defined by spreadsheet range A:1 to B:2 and vector $\vec{\mathbf{u}}$ is defined by spreadsheet range D:1 to D:2 then the Excel functions that give us the unit vector $\vec{\mathbf{v}}$ in Equation (47) above is...

$$\vec{\mathbf{v}} = \text{MMULT(MINVERSE(A1:B2),D1:D2)} \tag{49}$$

References

[1] Gary Schurman, *Modeling Exponential Arrival Times,* September, 2015.

Appendix

A. The solution to the following integral is...

$$\int\limits_0^s \lambda\,\mathrm{Exp}\left\{-\lambda\,t\right\}\delta t = \lambda\times-\frac{1}{\lambda}\,\mathrm{Exp}\left\{-\lambda\,t\right\}\Bigg[_0^s$$
$$= -\Bigg(\mathrm{Exp}\left\{-\lambda\,s\right\}-1\Bigg)$$
$$= 1-\mathrm{Exp}\left\{-\lambda\,s\right\} \tag{50}$$

B. The solution to the following integral is...

$$\int\limits_0^s \mathrm{Exp}\left\{\alpha\,(s-t)\right\}\mathrm{Exp}\left\{-\lambda\,t\right\}\delta t = \mathrm{Exp}\left\{\alpha\,s\right\}\int\limits_0^s \mathrm{Exp}\left\{-(\alpha+\lambda)\,t\right\}\delta t$$
$$= \mathrm{Exp}\left\{\alpha\,s\right\}(-\alpha-\lambda)^{-1}\mathrm{Exp}\left\{-(\alpha+\lambda)\,t\right\}\Bigg[_0^s$$
$$= \mathrm{Exp}\left\{\alpha\,s\right\}(-\alpha-\lambda)^{-1}\Bigg(\mathrm{Exp}\left\{-(\alpha+\lambda)\,s\right\}-1\Bigg)$$
$$= \mathrm{Exp}\left\{\alpha\,s\right\}(\alpha+\lambda)^{-1}\Bigg(1-\mathrm{Exp}\left\{-(\alpha+\lambda)\,s\right\}\Bigg) \tag{51}$$

C. The solution to the following integral is...

$$\int\limits_a^b \mathrm{Exp}\left\{\Delta\,t\right\}\mathrm{Exp}\left\{\alpha\,(T-t)\right\}\delta t = \mathrm{Exp}\left\{\alpha\,T\right\}\int\limits_a^b \mathrm{Exp}\left\{(\Delta-\alpha)\,t\right\}\delta t$$
$$= \mathrm{Exp}\left\{\alpha\,T\right\}(\Delta-\alpha)^{-1}\mathrm{Exp}\left\{(\Delta-\alpha)\,t\right\}\Bigg[_a^b$$
$$= \mathrm{Exp}\left\{\alpha\,T\right\}(\Delta-\alpha)^{-1}\Bigg(\mathrm{Exp}\left\{(\Delta-\alpha)\,b\right\}-\mathrm{Exp}\left\{(\Delta-\alpha)\,a\right\}\Bigg) \tag{52}$$